The stability of shearing motion in a rotating fluid

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(Received 18 March 1963 and in revised form 20 June 1963)

This paper is concerned with the stability of a parallel shear flow in an inviscid homogeneous unbounded rotating fluid. A sufficient condition for stability is obtained in terms of the dimensionless parameter $N = (\cos\phi)/S$, where ϕ is the angle between the wave-number $\mathbf{k}$ of the disturbance and the axis of rotation, and S is the Rossby number based on the thickness of the shear layer and the change in velocity across the layer. The condition is that infinitesimal disturbances are stable if either

$$N \geqslant \tfrac{1}{2}(1-\sin\theta) \quad \text{or} \quad N \leqslant -\tfrac{1}{2}(1+\sin\theta),$$

where θ is the angle between $\mathbf{k}$ and the direction of streaming. For a shear layer profile of the type $U = \tanh z$, the neutral curves are calculated for various Rossby numbers. These are compared to the stability of a shear layer in a stratified non-rotating fluid. The stability criterion for the large wave-numbers in a cylindrical shear layer is inferred from these results.

1. Introduction

Steady flows in rotating fluids are characterized by the appearance of shear layers parallel to the axis of rotation. Examples are the presence of forward wakes, and the experiments by Taylor (1921), in which thin sheets of dye were produced. The aim of this paper is to study the stability of these layers. Most earlier work on the stability of shear layers has been confined to non-rotating fluids, and in these problems Squire's theorem assures us that a study of two-dimensional disturbances is sufficient to determine the stability characteristics. A summary of this work has been given by Chandrasekhar (1961, ch. XI). Although the similarity between the stability characteristics of stratified fluids and rotating fluids was discussed by Lord Rayleigh (1916), most of the theory is confined to explanations of the Taylor instability of fluid between rotating concentric cylinders. However, in a recent paper, Howard & Gupta (1962) have now made this comparison for axisymmetric disturbances for both axial and swirl flows.

An attempt is now made to deal with the problem of a plane parallel flow in a homogeneous unbounded fluid, which has uniform rotation about a fixed axis. As the simplicity of Squire's theorem is destroyed by the presence of the Coriolis term, a three-dimensional perturbation problem has to be solved. The analogy with stratified flow is shown to hold for the plane case, and a sufficient condition for stability is established for a general plane parallel flow. When this condition is examined in detail for the continuously varying velocity profile, $\tilde{U} = U_0 \tanh(\tilde{z}/b)$, the curves of neutral stability are calculated for various combinations of the Rossby number and the direction of rotation. It is found that the effect of the rotation is to stabilize the fluid, particularly to disturbances along the axis of rotation.

Later, when the problem of a cylindrical shear layer is examined, it will be seen that the axial flow plays an important role in determining the stability characteristics. This is further evidence in support of the claim by Howard & Gupta that Chandrasekhar (1960) is incorrect in concluding that the swirl component alone determines the stability by Rayleigh's criterion, irrespective of the strength of the shear in the axial direction.

2. A converse of the Taylor–Proudman theorem

In this paper, it is assumed that the fluid is incompressible, inviscid, and in uniform rotation with angular velocity $\mathbf{\Omega}$ about an axis fixed in space. The rotating axes are chosen so that the initial velocity distribution $(\tilde{U}, \tilde{V}, \tilde{W})$ has only one non-zero component $\tilde{U}(\tilde{z})$, which is only a function of $\tilde{z}$.

The momentum equation, expressed in this rotating frame of reference, may be written as

$$\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + (\tilde{\mathbf{u}}.\tilde{\nabla})\,\tilde{\mathbf{u}} + 2\mathbf{\Omega}\times\tilde{\mathbf{u}} = -\tilde{\nabla}\,\tilde{p}', \tag{2.1}$$

where

$$\tilde{p}' = \tilde{p}/\rho - \tfrac{1}{2}(\mathbf{\Omega}\times\tilde{\mathbf{x}})^2 + \phi.$$

In these equations, $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tilde{t})$ represents the velocity field, $\tilde{p}$ the fluid pressure, and ϕ the potential of any applied conservative force field. The equation for the vorticity field $\tilde{\boldsymbol{\omega}}(\tilde{\mathbf{x}}, \tilde{t})$ may be found by taking the curl of (2.1), giving

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial \tilde{t}} + (\tilde{\mathbf{u}}.\tilde{\nabla})\,\tilde{\boldsymbol{\omega}} = (\tilde{\boldsymbol{\omega}} + 2\mathbf{\Omega}).\tilde{\nabla}\tilde{\mathbf{u}}. \tag{2.2}$$

Let us first consider whether there is a steady shear layer in a rotating fluid if the component of rotation perpendicular to the layer, Ω_z, is non-zero. The velocity and vorticity fields are given by

$$\bar{\tilde{\mathbf{u}}} = (\tilde{U}(\tilde{z}), 0, 0), \quad \bar{\tilde{\boldsymbol{\omega}}} = (0, d\tilde{U}/d\tilde{z},\ 0). \tag{2.3}$$

On substituting these values in equation (2.2), we have

$$2\Omega_z \frac{d\tilde{U}}{d\tilde{z}} = 0. \tag{2.4}$$

Clearly if $d\tilde{U}/d\tilde{z}$ is not zero, then $\Omega_z = 0$; the axis of rotation is necessarily parallel to the plane of the shear layer.

The Taylor–Proudman theorem states that all steady slow motions in a rotating inviscid fluid are two-dimensional. Taylor (1921) showed experimentally that when a small quantity of dye is introduced to a slightly disturbed rotating fluid, it spreads out into thin sheets parallel to the axis of rotation. Equation (2.4) shows that any steady shear flow must necessarily lie in planes parallel to the axis of rotation.

3. The perturbation equations

In parallel flow problems which do not involve uniform rotation, Squire (1933) has shown that the problem of the instability of three-dimensional disturbances is actually equivalent to a two-dimensional problem at a lower Reynolds number. However, when the fluid is rotating, it is not possible to use this theorem, as the vector product $2\boldsymbol{\Omega} \times \tilde{\mathbf{u}}$ in (2.1) does not allow a linear combination of two of the disturbance variables to satisfy the same equation. Consequently, a three-dimensional perturbation problem must be constructed.

There will be two basic scales, U_0 and b, defined by the shape of the shear layer, where $2U_0$ is the total change in the velocity across the layer, and b is a measure of the thickness of the layer. Using these two scales, dimensionless variables may be introduced as

$$\mathbf{u} = \tilde{\mathbf{u}}/U_0, \quad \mathbf{x} = \tilde{\mathbf{x}}/b, \quad \boldsymbol{\omega} = b\tilde{\boldsymbol{\omega}}/U_0, \quad t = U_0\tilde{t}/b, \tag{3.1}$$

and a dimensionless parameter, the Rossby number, $S = U_0/2b\Omega$.

Consider a small perturbation velocity field $\mathbf{u}' = (u, v, w)$ and vorticity field $\boldsymbol{\omega}' = (\xi, \eta, \zeta)$ to the steady initial fields described by (2.3) and (3.1). The rotation vector $\boldsymbol{\Omega} = \Omega\mathbf{n}$, where $\mathbf{n}$ is a unit vector lying in a plane parallel to the shear layer. When these variables are substituted in the vorticity equation (2.2), the linearized first-order perturbation equations are found to be

$$\frac{\partial \xi}{\partial t} + U\frac{\partial \xi}{\partial x} = U'\frac{\partial u}{\partial y} + \zeta U' + \frac{(\mathbf{n}.\nabla)}{S}u, \tag{3.2}$$

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} + wU'' = U'\frac{\partial v}{\partial y} + \frac{(\mathbf{n}.\nabla)}{S}v, \tag{3.3}$$

and

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} = U'\frac{\partial w}{\partial y} + \frac{(\mathbf{n}.\nabla)}{S}w, \tag{3.4}$$

where the primes represent differentiation with respect to z. In order to specify the problem completely, we must add the continuity equation, and the relationships between the velocity and vorticity fields,

$$\nabla.\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.5}$$

$$\zeta = \hat{\boldsymbol{\kappa}}.\nabla \times \mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{3.6}$$

and

$$-\nabla^2 w = \hat{\boldsymbol{\kappa}}.\nabla \times \boldsymbol{\omega} = \frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}, \tag{3.7}$$

where $\hat{\boldsymbol{\kappa}}$ is the unit vector in the z-direction. The variables u, v, ξ, η may be removed by combining equations (3.2) and (3.3), and using the relations (3.6) and (3.7), giving

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 w - U''\frac{\partial w}{\partial x} + \frac{(\mathbf{n}.\nabla)}{S}\zeta = 0,$$

which with (3.4) defines w and ζ. After eliminating ζ, the perturbation equation for w is

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\nabla^2 w-U''\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial w}{\partial x}+\frac{\mathbf{n}.\nabla}{S}\left(\frac{\mathbf{n}.\nabla}{S}+U'\frac{\partial}{\partial y}\right)w=0. \qquad (3.8)$$

As the variables x, y and t appear only in partial derivatives, we may seek sinusoidal solutions of the form

$$u(x,y,z,t)=\psi(z)\exp ik(x\cos\theta+y\sin\theta-ct), \qquad (3.9)$$

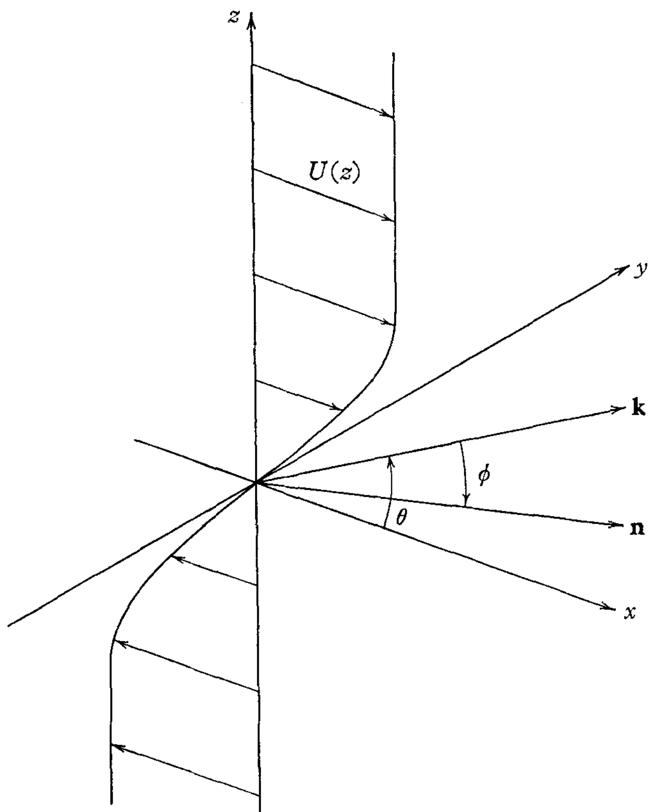

FIGURE 1. Notation.

where θ is the angle between the wave-number $\mathbf{k}$ and the direction of streaming. As a further simplification, the angle between the wave-number and the axis of rotation $\mathbf{n}$ will be called ϕ, as shown in figure 1. Substitute in equation (3.8) and we obtain

$$(U\cos\theta-c)^2(\psi''-k^2\psi)-U''\cos\theta(U\cos\theta-c)\psi+\frac{\cos\phi}{S}\left(\frac{\cos\phi}{S}+U'\sin\theta\right)\psi=0. \qquad (3.10)$$

The boundary conditions are chosen so that the perturbations are zero at large distances from the shear layer; that is

$$\psi\sim 0 \quad \text{as} \quad z\rightarrow\pm\infty. \qquad (3.11)$$

It is already clear from (3.10) that there will be a preferred direction in the solution of this problem, and that this direction is the axis of rotation. For, when the wave-number is perpendicular to this axis, the last term of (3.10) is zero, and the equation reduces to the inviscid Orr–Sommerfeld equation. Consequently, in this direction, the conditions for stability will be the same as the non-rotating case. On further examination of (3.10), it will be observed that the flow is really characterized by a new dimensionless parameter, $N = (\cos\phi)/S$, which has negative values when $\mathbf{k}$ and $\mathbf{n}$ do not lie in the same quadrant. N is the ratio between the inertial forces and the components of the Coriolis forces in the direction of the wave-number.

A sufficient condition for stability

A sufficient condition for stability may be derived using a standard method, but the result when θ is non-zero is rather unusual. Substitute $\psi = f(U\cos\theta - c)^{\frac{1}{2}}$ in (3.10), giving

$$\{(U\cos\theta - c)f'\}' - \tfrac{1}{2}fU''\cos\theta - k^2 f(U\cos\theta - c) - \frac{fU'^2\cos^2\theta}{4(U\cos\theta - c)} + \frac{N(N+U'\sin\theta)f}{(U\cos\theta - c)} = 0.$$

If this equation is multiplied by f^*, the complex conjugate of f, and integrated over the range of all z, we have

$$\int_{-\infty}^{\infty}(U\cos\theta - c)\{|f'|^2 + k^2|f|^2\}\,dz + \tfrac{1}{2}\int_{-\infty}^{\infty}|f|^2 U''\cos\theta\,dz + \int_{-\infty}^{\infty}\{\tfrac{1}{4}U'^2\cos^2\theta - N(N+U'\sin\theta)\}\left|\frac{f}{U\cos\theta - c}\right|^2(U\cos\theta - c^*)\,dz = 0. \tag{3.12}$$

Now if $c = c_r + ic_i$, where $c_i > 0$, there is instability, and the imaginary part of (3.12) is

$$\int_{-\infty}^{\infty}\{|f'|^2 + k^2|f|^2\}\,dz + \int_{-\infty}^{\infty}\{N(N+U'\sin\theta) - \tfrac{1}{4}U'^2\cos^2\theta\}\left|\frac{f}{U\cos\theta - c}\right|^2 dz = 0.$$

Clearly this is impossible if everywhere

$$N(N+U'\sin\theta) - \tfrac{1}{4}U'^2\cos^2\theta \geqslant 0. \tag{3.13}$$

Consequently this is a sufficient condition for the stability of a plane parallel flow with velocity profile $U(z)$, and it may be factorized into

$$\{N + \tfrac{1}{2}U'(1+\sin\theta)\}\{N - \tfrac{1}{2}U'(1-\sin\theta)\} \geqslant 0. \tag{3.14}$$

As no restriction has been placed on the velocity field $U(z)$ yet, (3.14) will also apply to other types of parallel flow, as well as the shear layer. This criterion will be examined more closely in the following sections, which deal with simplifications of (3.10).

4. A continuously varying velocity profile

In this section, the general problem is considerably simplified by working with a velocity profile that is continuously varying everywhere, and is defined by $U = \tanh z$. In this case, U' has a maximum value of unity, and its minimum value is zero, which allows (3.14) to be simplified. It can be shown that if everywhere N lies in the range

$$N \geqslant \tfrac{1}{2}(1-\sin\theta), \quad N \leqslant -\tfrac{1}{2}(1+\sin\theta), \tag{4.1}$$

then all disturbances are stable. This result is illustrated in figure 2. It shows that there is always a band of width one unit of N, where instability is possible for at least one wave-number. This band always lies within the range $|N| \leqslant 1$,

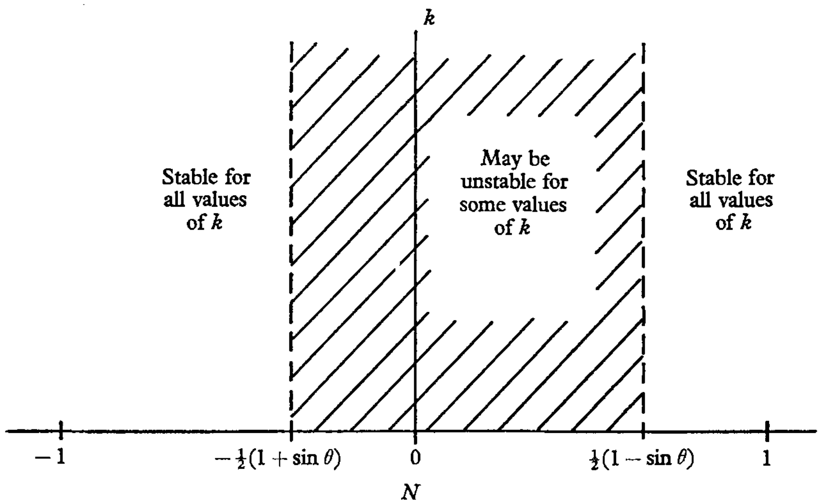

FIGURE 2. Diagrammatic representation of the sufficient condition for stability for the velocity profile $U = \tanh z$.

and its position is determined by θ, the direction of the wave-number vector. Moreover, for a given value of S, the thickness of the band depends on ϕ, the angle between $\mathbf{k}$ and the axis of rotation.

The solution for neutral waves

For the velocity profile, $U = \tanh z$, the eigenvalue problem defined by (3.10) and (3.11) may be solved, and the curves of neutral stability found, if we use the method developed by Drazin (1958) for the same velocity profile in a stratified non-rotating fluid. As (3.10) has space and time symmetry, we shall assume that $c = 0$ for a neutral wave, and then equation (3.10) reduces to

$$\psi'' - k^2\psi - \frac{U''}{U}\psi + \frac{N(N+U'\sin\theta)}{U^2\cos^2\theta}\psi = 0. \tag{4.2}$$

This is certainly true for a neutral wave when rotation is absent, as then the equation reduces to the Rayleigh equation, and we must have $c = U$ where

$U'' = 0$. For $U = \tanh z$, we have $U'' = 0$ when $U = 0$. Thus $c = 0$. The chief justification of this assumption lies in the fact that the resulting solution exists and matches the sufficient condition for stability obtained for arbitrary values of c_r. If we now change the independent variable to U and make use of the fact that $U = \tanh z$, we have

$$\frac{d}{dU}\left\{(1-U^2)\frac{d\psi}{dU}\right\}+\left\{2-\frac{k^2}{1-U^2}+\frac{N^2\sec^2\theta}{U^2(1-U^2)}+\frac{N\sin\theta\sec^2\theta}{U^2}\right\}\psi = 0, \quad (4.3)$$

and the boundary conditions become

$$\psi = 0 \quad \text{when} \quad U = \pm 1. \quad (4.4)$$

This equation has singular points at $U = 0, \pm 1$, and it may be verified that

$$\psi = (1-U^2)^\nu$$

is a regular solution near $U = \pm 1$, if

$$\nu = \tfrac{1}{2}(k^2 - N^2\sec^2\theta)^{\frac{1}{2}}. \quad (4.5)$$

Similarly, $\psi = U^\mu$ is a regular solution near $U = 0$, if

$$\mu = \tfrac{1}{2} + \tfrac{1}{2}\sec\theta\{(1-\sin\theta-2N)(1+\sin\theta+2N)\}^{\frac{1}{2}}. \quad (4.6)$$

We shall look for a solution of the form

$$\psi = U^\mu(1-U^2)^\nu\chi,$$

where χ is regular at the singular points of equation (4.3), and satisfies the following equation, derived from (4.3), using (4.5) and (4.6),

$$\frac{d^2\chi}{dU^2}+\left\{\frac{2\mu}{U}-\frac{2(2\nu+1)U}{1-U^2}\right\}\frac{d\chi}{dU}-\frac{(2\nu+\mu+2)(2\nu+\mu-1)}{(1-U^2)}\chi = 0.$$

A particular solution of this equation, which happens to be the solution we require, is given by

$$\chi = \text{const.}, \quad (2\nu+\mu+2)(2\nu+\mu-1) = 0.$$

As we have chosen the positive square roots in (4.5) and (4.6), in order to keep the solution regular at the singular points, $2\nu+\mu+2$ cannot be zero, and therefore

$$2\nu+\mu-1 = 0.$$

After substituting from (4.5) and (4.6), and simplifying, this becomes the equation for the eigenvalues

$$k^2(k^2-1)\cos^4\theta + 2k^2N\sin\theta\cos^2\theta + N^2 = 0. \quad (4.7)$$

The eigenfunctions for a neutral disturbance are given by

$$\psi = (\text{const.})(\operatorname{sech} z)^{2\nu}(\tanh z)^\mu,$$

where ν and μ are now

$$\nu = \tfrac{1}{2}k^2\cos^2\theta \pm \tfrac{1}{2}k\sin\theta(1-k^2\cos^2\theta)^{\frac{1}{2}}, \quad (4.8)$$

and

$$\mu = (1-k^2\cos^2\theta) \mp k\sin\theta(1-k^2\cos^2\theta)^{\frac{1}{2}}. \quad (4.9)$$

When $\theta = 0$, equations (4.7), (4.8) and (4.9) reduce to the results obtained by Drazin using a similar analysis, except that N^2 is replaced by the Richardson number for stratified flow. The similarity between rotating flow and stratified flow has been discussed by Howard & Gupta (1962) for the case of axisymmetric flow fields. The results shown above extend this similarity to plane parallel flows, but only for the case of wave-numbers parallel to the basic flow. In stratified fluids, these are the only disturbances considered, but as should already be clear, other directions are also just as important for rotating fluids.

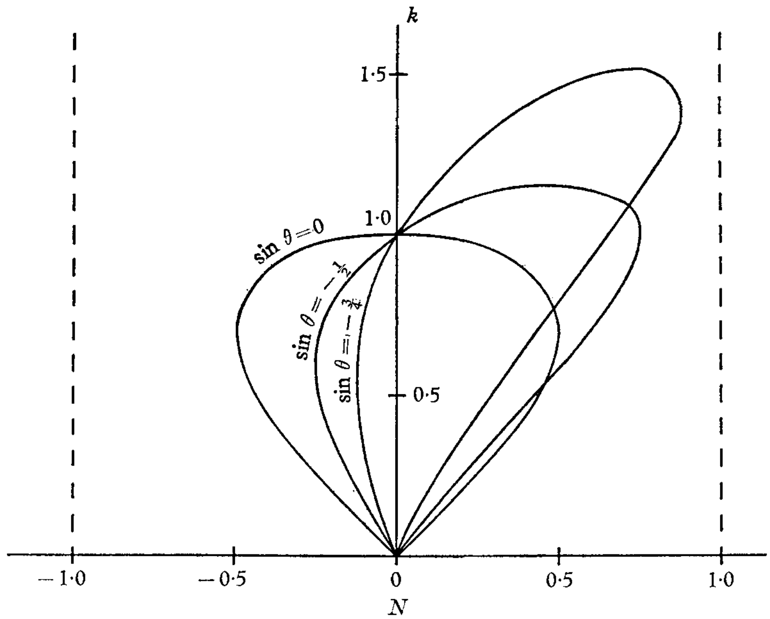

FIGURE 3. Neutral stability curves for various values of θ.

Equation (4.7) shows that there will be a different curve of neutral stability for each value of θ. A few of the members of this set of curves are shown in figure 3; there is a similar set for positive values of θ, which are the reflexions in the k-axis of those shown. These curves match exactly the sufficient conditions for stability as defined by figure 2, which are for arbitrary values of c_r. Using the preceding analysis, it is not possible to prove that these neutral curves separate the stable region of the (N, k)-plane from the unstable region. However, it may be shown that the full viscous equation for the perturbation reduces to the Orr–Sommerfeld equation for large S, and to equation (4.2) for large Reynolds number. From the asymptotic viscous theory of the Orr–Sommerfeld equation, it follows that the flow is stable for values (N, k) on the k-axis if $k > 1$, and unstable if $k < 1$. Hence the neutral curves (4.7) do form the boundary between stable and unstable regions in the non-rotating case ($N = 0$). Therefore it seems reasonable that they will still form the boundary in the rotating case, especially as they match the sufficient conditions for stability exactly. Although we can only prove that amplified disturbances exist when $N = 0$, it is unlikely that a small rotation will remove all the instabilities, and therefore it seems probable that other points inside the neutral loops represent unstable disturbances.

Interpretation of the stability curves

By inspection of the curves in figure 3, it may be seen that when **k** is perpendicular to the axis of rotation, N is zero for all values of S, and there is instability for all wave-numbers less than unity. Instability in other directions is not possible when the magnitude of N is greater than unity, and this requires $2\Omega\cos\phi > U_0/b$, a result that may be interpreted as follows. If the projection of

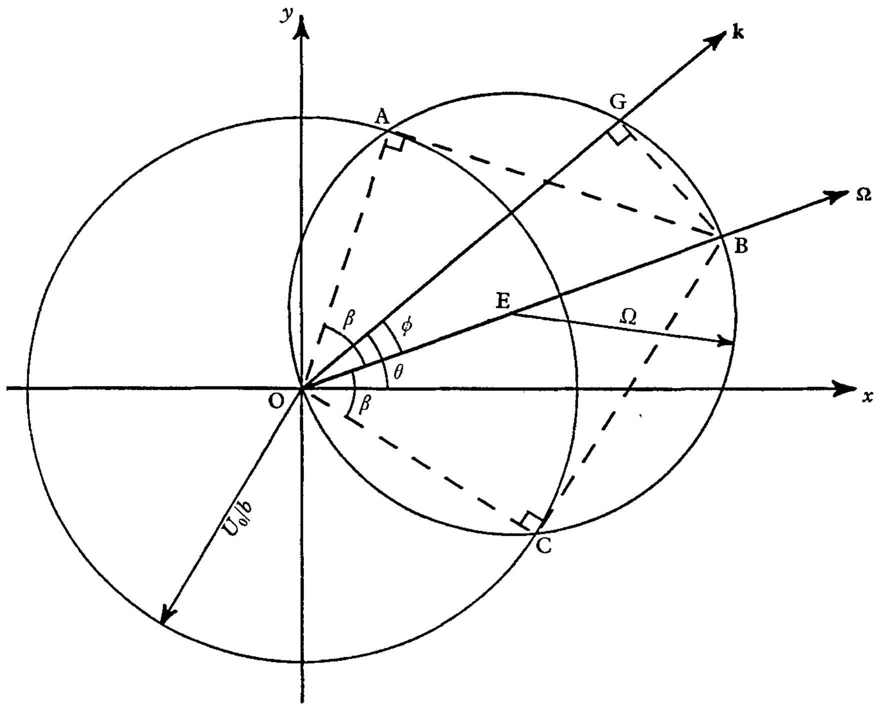

FIGURE 4. Definition of angle β. There is stability if k lies in the range $(\theta-\phi-\beta, \theta-\phi+\beta)$.

the mean vorticity, 2Ω, on to the wave-number vector is greater than the maximum vorticity due to the shearing motion, there will be stability. This inequality leads to a sufficient condition that there will be stability in a well-defined pencil of directions of **k** for each (θ, ϕ) pair. In figure 4, the circle with centre at O has radius U_0/b, and the circle with centre E has radius Ω, where OE is the vector $\mathbf{\Omega}$. For prescribed values of θ and ϕ, OG has magnitude $2\Omega\cos\phi$. This will be larger than U_0/b if G lies on the arc ABC, and therefore all positions of **k** lying between OA and OC will be stable directions; that is for $\phi < \beta$. From the geometry of the figure

$$\cos\beta = \frac{\mathrm{OA}}{\mathrm{OB}} = \frac{U_0/b}{2\Omega} = S,$$

and the range of stable directions is between $\theta-\phi-\beta$ and $\theta-\phi+\beta$. As this is only a sufficient condition, it is possible that other directions will also be stable.

An important distinction between the stability characteristics of rotating and non-rotating fluids is already clear. In a non-rotating fluid, the conditions for stability are the same in all directions, whereas in a rotating fluid, some directions may be stabilized, whilst other directions remain unstable.

The neutral stability curves in the (k, θ)-plane

Further interpretation of figure 3 is rather complicated by the fact that it involves four variables, but if the direction of rotation is specified, ϕ may be expressed as a function of θ, and then it is possible to draw neutral stability curves in the (k, θ)-plane, for various values of the Rossby number. Figure 5 consists of five graphs which describe the neutral curves for five different positions of the axis of rotation. The interior of each loop is the unstable region. These graphs may also be used to describe the remaining three positions at intervals of $\pi/4$ radians. For $\mathbf{\Omega}$ in the directions $\theta = 3\pi/4, \pi$, and $5\pi/4$, the reflexion in the line $\theta = 0$ of the curves of figures 5 (*b*), (*c*) and (*d*) give respectively the appropriate graphs. These graphs may be used to enlarge the pencil of stable directions defined in figure 4 by the angle AOC. Thus, it becomes clear that, as S tends to zero, more directions become stabilized, until in the limit the only direction in which disturbances are unstable is perpendicular to the axis of rotation. As S approaches zero, the fluid motion becomes dominated by the rotation, whose chief effect is to stabilize the fluid, particularly to disturbances along its own axis. It seems appropriate to point out here that for a non-rotating fluid, the polar neutral stability curve is the circle $k = 1$, together with the line $\theta = \pi/2$; this line must be included as there is no component of the shearing motion in this direction, and therefore the wave-speed c is zero for all wave-numbers. It will be seen that as S increases, the five neutral curves approach this limiting curve. In the next section, it is shown that the effects of viscous damping become important for large wave-numbers, and therefore the infinite parts of the curves in figure 5 may be cut off at a finite value of k.

On close inspection of figure 5, it appears that all the neutral loops lie within the boundaries defined by two straight lines, parallel to the y-axis, drawn through the points $k = 1$ on the x-axis. That these are the enveloping lines becomes clear after some rearrangement and differentiation of (4.7). Their equation is

$$k_x = k \cos \theta = 1.$$

Therefore there is always stability for wave-numbers whose component, in the direction of the basic streaming, is greater than unity. This would be both a necessary and sufficient condition for stability if viscosity had no effect on the behaviour of large wave-numbers.

The considerable variation between figure 5 (*a*) and 5 (*e*) is rather surprising as the only physical difference is the reversal of the direction of uniform rotation. However, the fundamental difference is that in case (*a*) the rotation assists the shearing motion, whereas in case (*e*) it opposes this motion. This effect is dealt with in detail in § 6, where it is explained in terms of Rayleigh's criterion for axisymmetric flow.

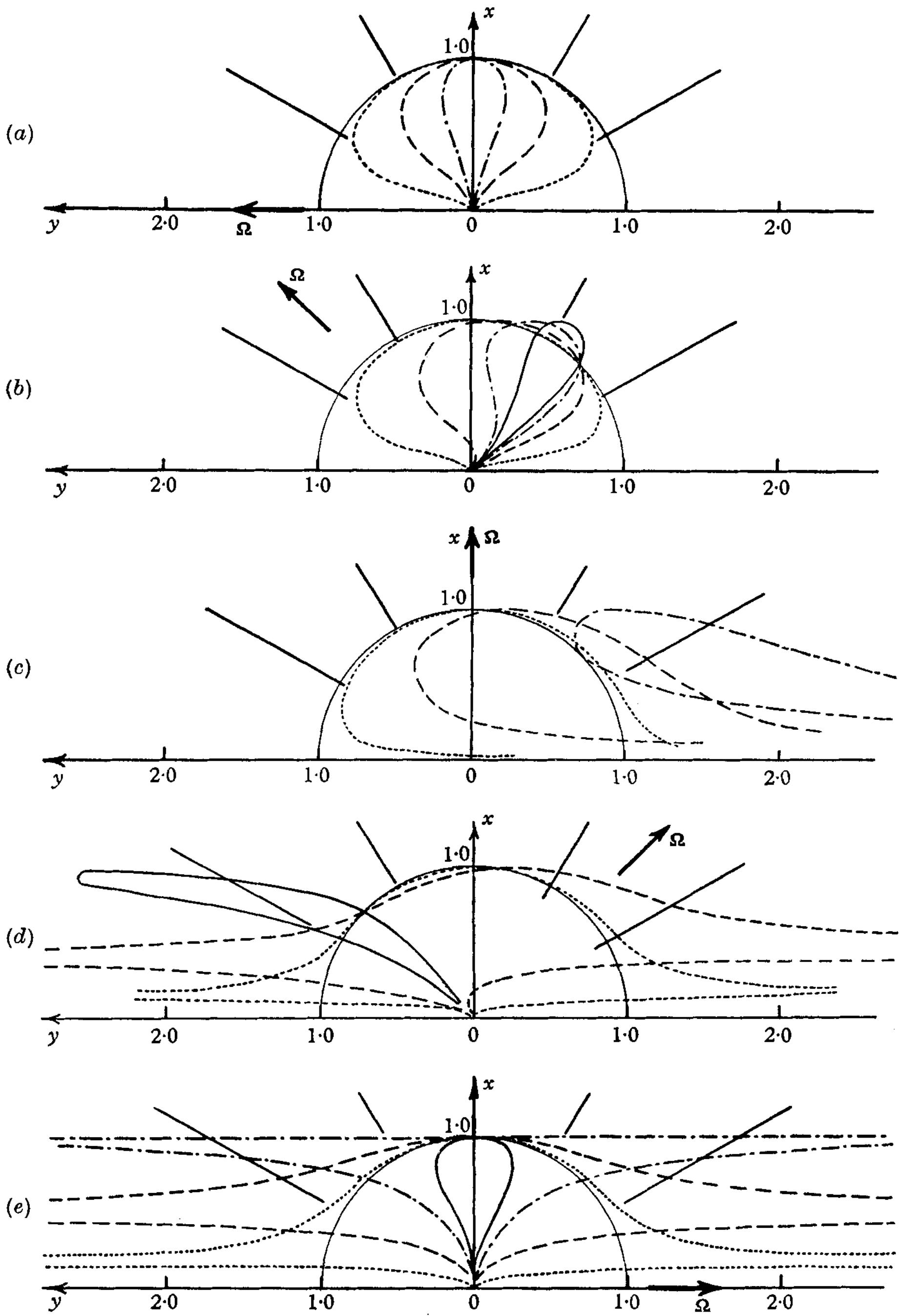

FIGURE 5. Neutral stability curves in the (k, θ)-plane for various values of S, and for $\mathbf{\Omega}$ in the direction of (*a*) $\theta = 90°$, (*b*) $\theta = 45°$, (*c*) $\theta = 0°$, (*d*) $\theta = -45°$, and (*e*) $\theta = -90°$. The values of S are represented by: $\cdots$, $S = 32$; ---, $S = 4$; -·-·, $S = 1$; and ——, $S = \frac{1}{2}$. The numerical scale represents the magnitude of k measured radially from the origin.

The role of the total vorticity

For the particular velocity profile under consideration, the Rossby number is essentially a balance between the mean vorticity, $2\Omega\mathbf{n}$, and the vorticity, $(U_0/b)\,\hat{\mathbf{j}}$, due to the shearing motion across the plane of symmetry of the layer, where $\hat{\mathbf{j}}$ is the unit vector along the y-axis. The total vorticity is $\hat{\boldsymbol{\omega}} = 2\Omega\mathbf{n} + (U_0/b)\,\hat{\mathbf{j}}$, and its direction is $(\mathbf{n}+S\hat{\mathbf{j}})$. If we assume that the most stable direction, for fixed S, is given by the direction of the total vorticity, rather than the mean vorticity, and then examine figure 5, we find that this assumption fits in well with the results displayed. It is best illustrated by the curves in figures 5 (b) and (c). It explains why the first directions to become stable, for large S, are near the y-axis; and why the last directions to remain unstable for small S are near the direction that is perpendicular to $\mathbf{n}$.

It is possible that this assumption may be explained in terms of the stretching of vortex lines. The effect of the perturbation is to deform the straight vortex lines into helices with axes parallel to $\hat{\boldsymbol{\omega}}$ and wavelength $(2\pi\sec\gamma)/k$, where γ is the angle between $\mathbf{k}$ and $\hat{\boldsymbol{\omega}}$. When $\mathbf{k}$ is parallel to $\hat{\boldsymbol{\omega}}$, the helices are more compressed and hence the vortex lines are more stretched than for any other position of $\mathbf{k}$. The vortex force $(\mathbf{u}\times\hat{\boldsymbol{\omega}})$ acts as the restoring force, as at every point of the vortex line it is directed towards the axis of the helix. Its maximum value occurs when $\mathbf{k}$ is parallel to $\hat{\boldsymbol{\omega}}$. Therefore larger disturbing forces are required to produce instability in the direction of $\hat{\boldsymbol{\omega}}$, and consequently it is the most stable direction.

5. The effect of viscosity on large wave-number disturbances

If we include the viscous terms in the vorticity equation, we must include the term $\nu\tilde{\nabla}^2\tilde{\boldsymbol{\omega}}$ in equation (2.2). After introducing the dimensionless Reynolds number, $R = U_0 b/\nu$, the effect of viscosity is obtained by replacing the factor

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right) \quad \text{by} \quad \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+\frac{1}{R}\nabla^2\right)$$

in (3.8), thereby making the equation sixth order. After transforming to wave-number space, the viscous equation becomes

$$(U\cos\theta-c)^2(D^2-k^2)\,\psi-(D^2U)\cos\theta\,(U\cos\theta-c)\,\psi+\frac{\cos\phi}{S}\left\{\frac{\cos\phi}{S}+(DU)\sin\theta\right\}\psi$$
$$=\frac{(D^2-k^2)^3}{(kR)^2}\,\psi-\frac{2i}{kR}(DU)\cos\theta(D^2-k^2)\,D\psi-\frac{2i}{kR}(U\cos\theta-c)\,(D^2-k^2)^2\,\psi, \tag{5.1}$$

where $D\equiv d/dz$; for large R, this equation reduces to (3.10).

We are only interested in finding when the viscous terms become as important as the terms in (3.10) for large wave-numbers. If we adopt the same method as in the inviscid analysis, and allow c to be zero for a neutral wave, the dominant terms in (5.1) for large wave-number are

$$k^2U^2\psi\cos^2\theta,\quad \frac{k^6\psi}{(kR)^2},\quad \frac{2ik^4U\psi\cos\theta}{kR}. \tag{5.2}$$

From the shape of the curves in figure 5, we see that $k\cos\theta$ is $O(1)$ for large k, and therefore the first term of (5.2) is $O(1)$. Thus the viscous terms will be important if $k \sim O(R^{\frac{1}{2}})$.

For plane shearing motion, Esch (1957) showed that large wave-numbers are always stable for all values of the Reynolds number. Therefore, under these conditions, viscosity cannot possibly act as a destabilizer. So we may now round off all the parts of the neutral curves where k becomes large, as in these regions viscosity plays an important role, and any small disturbances will be damped.

6. Applications to the theory of axisymmetric vortex sheets

The velocity profile used in § 4 is defined in dimensional variables as

$$\tilde{U} = U_0 \tanh(\tilde{z}/b).$$

In the limit as b tends to zero, this profile has the plane vortex sheet as its asymptotic form. Therefore a cylindrical layer of vorticity may be represented by a modification of this profile if the radius of the cylinder R_0 is large compared with the thickness b of the layer. If we can show that the equations for small perturbations of the cylindrical layer and the plane layer are identical when $R_0 \gg b$, then the stability curves obtained in § 4 may be used in this section.

At first, we shall consider the more general axisymmetric problem, in which the z-axis is now chosen as the axis of rotation to conform with standard notation. The other cylindrical polar co-ordinates are r and α. The basic velocity field is defined by an axial component $W(r)$, and a swirl component $V(r)$. With the understanding that ∇ now represents the gradient in cylindrical polars, the notation is the same as that used in § 3. After substituting in the non-dimensional form of (2.2), we have

$$\frac{\partial \xi}{\partial t}+\frac{V}{r}\frac{\partial \xi}{\partial \alpha}+W\frac{\partial \xi}{\partial z} = -\frac{W'}{r}\frac{\partial u}{\partial \alpha}+\left(\frac{V}{r}+V'\right)\frac{\partial u}{\partial z}+\frac{1}{S}\frac{\partial u}{\partial z}, \tag{6.1}$$

$$\frac{\partial \eta}{\partial t}+\frac{V}{r}\frac{\partial \eta}{\partial \alpha}+W\frac{\partial \eta}{\partial z}-uW'' = -\frac{W'}{r}\frac{\partial v}{\partial \alpha}+\left(\frac{V}{r}+V'\right)\frac{\partial v}{\partial z}+\xi V'+\frac{1}{S}\frac{\partial v}{\partial z}, \tag{6.2}$$

$$\frac{\partial \zeta}{\partial t}+\frac{V}{r}\frac{\partial \zeta}{\partial \alpha}+W\frac{\partial \zeta}{\partial z}+u\left(V''+\frac{V'}{r}-\frac{V}{r^2}\right) = -\frac{W'}{r}\frac{\partial w}{\partial \alpha}+\left(\frac{V}{r}+V'\right)\frac{\partial w}{\partial z}+\xi W'+\frac{1}{S}\frac{\partial w}{\partial z}. \tag{6.3}$$

However, the equations corresponding with (3.5) to (3.7) are slightly different, due to the effect of the non-Cartesian co-ordinate system, and they become

$$\nabla . \mathbf{u} = \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \alpha}+\frac{\partial w}{\partial z} = 0, \tag{6.4}$$

$$\xi = (\nabla \times \mathbf{u})_r = \frac{1}{r}\frac{\partial w}{\partial \alpha}-\frac{\partial v}{\partial z}, \tag{6.5}$$

$$D^{*2}u = \frac{\partial \eta}{\partial z}-\frac{1}{r}\frac{\partial \zeta}{\partial \alpha} = \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \alpha^2}-\frac{2}{r^2}\frac{\partial v}{\partial \alpha}+\frac{\partial^2 u}{\partial z^2}. \tag{6.6}$$

By a suitable combination of (6.2) and (6.3) in order that (6.5) and (6.6) may be usefully employed, a second equation for u and ξ is obtained,

$$\left(\frac{\partial}{\partial t}+\frac{V}{r}\frac{\partial}{\partial \alpha}+W\frac{\partial}{\partial z}\right)D^{*2}u-W''\frac{\partial u}{\partial z}-\left(V''+\frac{V'}{r}-\frac{V}{r^2}\right)\frac{1}{r}\frac{\partial u}{\partial \alpha}+\frac{1}{S}\frac{\partial \xi}{\partial z}=0. \quad (6.7)$$

We shall look for sinusoidal solutions of the form

$$(u,v)(r,\alpha,z,t)=(\psi,\bar{v})(r)\exp ik(r\alpha\sin\phi+z\cos\phi-ct), \quad (6.8)$$

where ϕ is the angle between the wave-number and the axis of rotation. In order to preserve our previous notation, we shall define θ as the angle between the wave-number and the direction of the resultant velocity $\overline{U}$, where $\overline{U}=(V^2+W^2)^{\frac{1}{2}}$, and then

$$V\sin\phi+W\cos\phi=\overline{U}\cos\theta,$$
$$V\cos\phi-W\sin\phi=\overline{U}\sin\theta.$$

After substituting these relations and (6.8) into equations (6.1) and (6.7), we may eliminate ξ, and obtain the equation for the perturbation ψ:

$$(\overline{U}\cos\theta-c)^2D^{*2}\psi-\overline{U}''\cos\theta(\overline{U}\cos\theta-c)\psi+\left(\frac{V}{r^2}-\frac{V'}{r}\right)(\overline{U}\cos\theta-c)\psi\sin\phi$$
$$+\frac{\cos\phi}{S}\left(\frac{\cos\phi}{S}+\overline{U}'\sin\theta+\frac{V}{r}\cos\phi\right)\psi=0, \quad (6.9)$$

where
$$D^{*2}\psi=\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}-\frac{\psi}{r^2}-k^2\psi-\frac{2i}{r}k\bar{v}\sin\phi.$$

For a cylindrical shear layer with large dimensionless radius R_0/b, and unit thickness, if the displacement of the centre of the layer always lies within the range $\{(R_0/b)-1,(R_0/b)+1\}$, the radial position of the layer, r, is $O(R_0/b)$ and dr is $O(1)$. Hence the first and fourth terms of $D^{*2}\psi$ are dominant for $R_0 \gg b$, and (6.9) reduces to

$$(\overline{U}\cos\theta-c)^2\left(\frac{d^2\psi}{dr^2}-k^2\psi\right)-\overline{U}''\cos\theta(\overline{U}\cos\theta-c)\psi+\frac{\cos\phi}{S}\left(\frac{\cos\phi}{S}+\overline{U}'\sin\theta\right)\psi=0. \quad (6.10)$$

This is identical with (3.10), and therefore the calculations relating to the plane shear layer may now be used for the cylindrical shear layer, provided $R_0 \gg b$.

In general, only two special cases are interesting, namely when either the axial or the swirl component of the velocity field is zero. In the latter case, the flow pattern is similar to a wake, whereas by suitable choice of $V(r)$, the former case may represent the motion between two rotating concentric cylinders.

Rayleigh's criterion for axisymmetric disturbances

We choose the basic flow field,

$$\tilde{V}_1(\tilde{r})=-V_0-V_0\tanh\{(\tilde{r}-R_0)/b\}, \quad \tilde{W}(\tilde{r})=0, \quad (6.11)$$

where $\tilde{V}_1$ is in dimensional variables; this represents a cylindrical shear layer near $\tilde{r}=R_0$ between a region with zero velocity and a region with velocity

$-2V_0$. For axisymmetric disturbances, this configuration corresponds to the plane case when $\theta = \pi/2$, $\phi = \pi$, and it is for these values that the sufficient condition for stability, represented in figure 2, has an unusual position. This may be explained by an appeal to Rayleigh's criterion for the instability of revolving fluids. He stated that if the circulation increases monotonically outwards, there is stability. The velocity field produced by the uniform rotation is $\tilde{V}_2(\tilde{r}) = \Omega\tilde{r}$, which opposes the shearing motion $\tilde{V}_1(\tilde{r})$, and therefore the total velocity field is

$$\begin{aligned}\tilde{V}_* &= \tilde{V}_2 - \tilde{V}_1 \\ &= \Omega\tilde{r} - V_0 - V_0 \tanh\{(\tilde{r} - R_0)/b\}.\end{aligned}$$

Hence
$$\left[\frac{d}{d\tilde{r}}(\tilde{r}\tilde{V}_*)\right]_{\tilde{r}=R_0} = 2\Omega R_0 - V_0 - V_0\frac{R_0}{b},$$

and for stability this must be non-negative. That is

$$2\Omega \geqslant \frac{V_0}{b}\left(1 + \frac{b}{R_0}\right) \approx \frac{V_0}{b} \quad \text{for} \quad b \ll R_0.$$

This shows that for stability, $S \leqslant 1$, and therefore as $\phi = \pi$, it requires that $N \leqslant -1$.

However, if the direction of the shearing flow is reversed, it is now assisted by the velocity field due to the rotation, and

$$\tilde{V}_* = \Omega\tilde{r} + V_0 + V_0 \tanh\{(\tilde{r} - R_0)/b\}.$$

For axisymmetric disturbances, this is analogous to the plane case when $\theta = \pi/2$, $\phi = 0$. The gradient of the circulation, at $\tilde{r} = R_0$, is given by

$$\left[\frac{d}{d\tilde{r}}(\tilde{r}\tilde{V}_*)\right]_{\tilde{r}=R_0} = 2\Omega R_0 + V_0 + V_0 R_0/b,$$

which is non-negative for all values of S, and therefore, as $\phi = 0$, there is stability if $N \geqslant 0$. These results are consistent with the stability condition (4.1) with $\theta = \pi/2$ (see figure 2).

These two different cases are also represented in figure 5(*e*) and (*a*) respectively, but it will be seen that if axial flow is also present, the conditions for stability are altered. In particular, if

$$\tilde{V}(\tilde{r}) = 0, \quad \tilde{W}(\tilde{r}) = W_0 + W_0 \tanh\{(R_0 - \tilde{r})/b\}, \tag{6.12}$$

the range of stability for axisymmetric disturbances is given by figure 3 when $\theta = \phi = 0$. It shows that instability only occurs for wave-numbers $0 < k \leqslant 1$, when $|N| = S^{-1} \leqslant \frac{1}{2}$, and that the most unstable wave-number is $k = 2^{-\frac{1}{2}}$. For basic distributions of velocity between (6.11) and (6.12), the stability criteria gradually change from one extreme to the other. Clearly this supports the claim made by Howard & Gupta (1962), that Chandrasekhar (1960) is incorrect in concluding that the swirl component alone determines the stability by Rayleigh's criterion, irrespective of the strength of the shear in the axial direction.

Asymmetric disturbances

Not very much may be inferred from the plane case for waves that are not axially symmetrical, as it is difficult to interpret the meaning of k_α. This wave-number is proportional to $(b/R_0)\,n$, where n is the number of waves around the cylindrical layer. If $n \neq 0$, in order to satisfy the condition that the layer should locally approximate to a plane, it is essential that $n \gg 1$. Figure 5(c) shows that for the shearing flow defined by (6.12), all disturbances with $n \neq 0$ are stable for all finite Rossby numbers, but only neutrally stable for infinite Rossby number. Figure 5(a) shows that for the velocity field defined by (6.11) there is a finite number of values of n representing unstable disturbances. This range of values for n is defined by the condition that all values of k_α in the range (0, 1) represent unstable waves, and they are independent of the Rossby number. Thus the only effect of the rotation on the stability of asymmetric disturbances is for 'wake' flows, where stable modes replace any neutral modes that may exist in non-rotating fluids.

It is a pleasure to record my thanks to Dr O. M. Phillips for his continued interest and advice during the course of this investigation.

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